

CORRELATIONS OF THE MOEBIUS AND LIOUVILLE FUNCTIONS AND THE TWIN PRIME CONJECTURE¹

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ABSTRACT. In this note we describe weight functions that exhibit a transitional behavior between weak and strong correlation with the Liouville function. We also describe a binary problem which may be considered as an interpolation between Chowla's conjecture for two-point correlations of the Möbius function and the twin prime conjecture, in view of recent parity breaking results of K. Matomäki, M. Radziwiłł and T. Tao.

1. Introduction. The Chowla conjecture for the Liouville function $\lambda(n)$ asserts that

$$\sum_{n \leq N} \lambda^{e_1}(n + h_1) \dots \lambda^{e_k}(n + h_k) = o(N)$$

as $N \rightarrow \infty$ for any fixed integers h_1, \dots, h_k with $h_i \neq h_j$ for $i \neq j$ and with at least one odd exponent e_i . In [Tao15] Tao proved the following logarithmically averaged version of Chowla's conjecture for two-point correlations:

Theorem 1 (T. Tao). *Let a_1, a_2 be natural numbers, and let b_1, b_2 be integers such that $a_1 b_2 - a_2 b_1 \neq 0$. Let $1 \leq h(x) \leq x$ be a quantity depending on x that goes to infinity as $x \rightarrow \infty$. Then one has*

$$\sum_{x/h(x) < n \leq x} \frac{\lambda(a_1 n + b_1) \lambda(a_2 n + b_2)}{n} = o(\log h(x))$$

as $x \rightarrow \infty$.

The twin prime conjecture is the assertion that

$$H_1 := \liminf_{n \rightarrow \infty} (p_{n+1} - p_n) = 2.$$

In 2004 Goldston, Pintz and Yıldırım [GPY09] established $H_1 \leq 16$ on the Elliott–Halberstam conjecture [EH70]. The breakthrough paper of Zhang [Zh13] shows that $H_1 \leq 70\,000\,000$ unconditionally. Subsequent improvements [Pm14] lowered this bound to $H_1 \leq 246$. From the generalized Elliott–Halberstam hypothesis (see Conjecture 1) Polymath8b project has managed to reach $H_1 \leq 6$. The paper [Pm14, Section 8] also provides a heuristic argument that the parity problem prohibits a sieve-theoretic proof of the twin prime conjecture.

Now we state a version of the generalized Elliott–Halberstam conjecture from [Pm14]. The Dirichlet convolution $\alpha \star \beta: \mathbb{N} \rightarrow \mathbb{C}$ of two arithmetic functions $\alpha, \beta: \mathbb{N} \rightarrow \mathbb{C}$ is defined as

$$\alpha \star \beta(n) := \sum_{d|n} \alpha(d) \beta\left(\frac{n}{d}\right) = \sum_{ab=n} \alpha(a) \beta(b).$$

For any function $\alpha: \mathbb{N} \rightarrow \mathbb{C}$ with finite support (that is, α is non-zero only on a finite set) and any primitive congruence class $a \pmod{q}$, we define the (signed) *discrepancy* $\Delta(\alpha; a \pmod{q})$ to be the quantity

$$\Delta(\alpha; a \pmod{q}) := \sum_{n \equiv a \pmod{q}} \alpha(n) - \frac{1}{\varphi(q)} \sum_{(n,q)=1} \alpha(n).$$

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Conjecture 1 (GEH (D)). *Let $D = x^{1-\varepsilon(x)}$ for some function $\varepsilon(x) > 0$ and $x/D \leq N, M \leq D$, be such that $NM \sim x$, and let α, β be coefficient sequences at scale N, M . Then*

$$\sum_{q \lesssim D} \sup_{a \in (\mathbb{Z}/q\mathbb{Z})^\times} |\Delta(\alpha \star \beta; a(q))| \ll x \log^{-A} x$$

for any fixed $A > 0$.

The generalized Elliott–Halberstam conjecture (GEH) will refer to the assertion that GEH (D) holds for all $D = x^{1-\varepsilon(x)}$ with $\frac{1}{(\log x)^{1-\delta}} \leq \varepsilon(x) < 1$ and some fixed $0 < \delta < 1$. We remark that from the results of [FGHM91] it follows that Conjecture 1 fails with $\varepsilon(x) \leq \frac{(A-\delta)(\log \log x)^2}{(\log x)(\log \log \log x)}$.

Let $\lambda(n)$ be the Liouville function, $\mu(n)$ be the Möbius function, $\tau_r(n)$ be the r -th divisor function for real r , that is the multiplicative function given for all primes p and all $m \geq 1$ by

$$\tau_r(p^m) = \frac{\Gamma(r+m)}{\Gamma(r)m!}.$$

For a positive integer n and $y \geq 2$ let $\omega_+(n, y)$ be the number of prime divisors of n which are $\geq y$ and $\omega_-(n, y)$ be the number of prime divisors of n which are $< y$. On squarefree integers n define the multiplicative function $\tau_{\kappa_1, \kappa_2}^\pm(n, y)$ by

$$\tau_{\kappa_1, \kappa_2}^\pm(n, y) := \kappa_1^{\omega_-(n, y)} \kappa_2^{\omega_+(n, y)}.$$

Let $1_S(n)$ be the indicator function of S . Suppose that (h_1, h_2) is a fixed admissible 2-tuple.

Another assumption of this note is based on the relationship between the function $\tau_{\kappa_1, \kappa_2}^\pm(n, y)$ with some fixed $0 \leq \kappa_1 < \kappa_2$ and the integers left unsieved when sieving with the primes less than $y = \exp((\log x)^\delta)$ for some fixed $0 < \delta < 1$. More precisely, we have the following relevant results.

Theorem 2 (K. Alladi [All82]). *Define*

$$S(x, y) := \{n \leq x : \text{least prime divisor of } n \text{ is } \geq y\},$$

$$\varphi_t(x, y) := \#\{n \in S(x, y) : \omega(n) - \log u < t\sqrt{\log u}\},$$

where $u = \log x / \log y$. Then for $2 \leq y \leq x$

$$\left| \varphi_t(x, y) / |S(x, y)| - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^t e^{-v^2/2} dv \right| \ll \frac{1}{\sqrt{\log u}}.$$

Theorem 3 (A. Selberg). *Let $\pi_k(x)$ be the number of positive integers less than x which have exactly k distinct prime divisors (not counting multiplicity). Then for $k = O(\log \log x)$*

$$\pi_k(x) \sim F\left(\frac{k-1}{\log \log x}\right) \frac{x}{\log x} \frac{(\log \log x)^{k-1}}{(k-1)!},$$

where

$$F(z) = \frac{1}{\Gamma(z+1)} \prod_p \left(1 + \frac{z}{p-1}\right) \left(1 - \frac{1}{p}\right)^z$$

and for $0 < \kappa \leq 1$

$$\sum_{n \leq x} \tau_\kappa(n) \sim c(\kappa) x (\log x)^{\kappa-1}.$$

In view of this results, one may consider the weight $\tau_\kappa(n)$ for $0 < \kappa < 1$ as a kind of preliminary sieving, since this weight is mostly concentrated on numbers which have $\kappa \log \log x$ prime divisors. Next we can make actual sieving in the weighted sum

$$\sum_{N/2 < n \leq N} \lambda(n+h_1)\lambda(n+h_2)\tau_\kappa(n+h_1)\tau_\kappa(n+h_2)$$

with the primes less than $y = \exp((\log x)^\delta)$. This construction seems to be more viable when κ is close to 1 and δ is close to 0.

More generally, we can make the following

Conjecture 2 (Generalized Chowla's conjecture). *For any fixed $0 \leq \kappa_1 < \kappa_2$ and $y = \exp((\log x)^\delta)$ with some fixed $0 < \delta < 1$ we have*

$$\sum_{\frac{N}{2} \leq n \leq N} \mu(n+h_1)\mu(n+h_2)\tau_{\kappa_1, \kappa_2}^\pm(n+h_1, y)\tau_{\kappa_1, \kappa_2}^\pm(n+h_2, y) = o\left(\sum_{\frac{N}{2} \leq n \leq N} \tau_{\kappa_1, \kappa_2}^\pm(n+h_1, y)\tau_{\kappa_1, \kappa_2}^\pm(n+h_2, y)\right).$$

In the case $\kappa_1 = \kappa_2 = 1$ this is the original Chowla's conjecture for two-point correlations of the Möbius function. And in the case of any $0 \leq \kappa_1 < \kappa_2$ this conjecture essentially implies the twin prime conjecture in view of the results of Bombieri (see [FI10, Chapter 16]). Obviously, if we had the case $\kappa_1 = \kappa_2 = 1$ then by continuity we could perturb κ_1, κ_2, y to get into the region $\kappa_1 < \kappa_2$.

Next, define the weight $\alpha_\kappa(n)$ for $0 \leq \kappa \leq 1$ by

$$\alpha_\kappa(n) := \sum_{d|n} \mu(d)\tau_{1+\kappa}\left(\frac{n}{d}\right)\left(\frac{\log n}{\log N} - \frac{\log d}{\log N}\right).$$

The result of this note is

Theorem 4. *We have*

$$\sum_{\frac{N}{2} \leq n \leq N} \lambda(n)\alpha_1(n) = o\left(\sum_{\frac{N}{2} \leq n \leq N} \alpha_1(n)\right)$$

but

$$\sum_{\frac{N}{2} \leq n \leq N} \lambda(n)\alpha_0(n) = -\left(\sum_{\frac{N}{2} \leq n \leq N} \alpha_0(n)\right)$$

with the sums on the right-hand sides being nonzero for all sufficiently large values of N . For two-point correlations, we have unconditionally

$$\sum_{\frac{N}{h(N)} \leq n \leq N} \frac{\lambda(n+h_1)\lambda(n+h_2)\alpha_1(n+h_1)\alpha_1(n+h_2)}{n} = o\left(\sum_{\frac{N}{h(N)} \leq n \leq N} \frac{\alpha_1(n+h_1)\alpha_1(n+h_2)}{n}\right)$$

and assuming Conjectures 1–2

$$\sum_{\frac{N}{2} \leq n \leq N} \lambda(n+h_1)\lambda(n+h_2)\alpha_0(n+h_1)\alpha_0(n+h_2) = \sum_{\frac{N}{2} \leq n \leq N} \alpha_0(n+h_1)\alpha_0(n+h_2)$$

with the sum on the right-hand side being nonzero. The latter claim implies the twin prime conjecture.

Lemma 1 (Leibnitz-type identity). *Let $L(n) = \log n$ and \star be the Dirichlet convolution. Then*

$$L \times (\mu \star 1 \star \tau_\kappa) = (L\mu) \star 1 \star \tau_\kappa + \mu \star (L\tau_{1+\kappa}).$$

Proof. The lemma follows by differentiating the generating Dirichlet series.

2. Proof of Theorem 4. By Lemma 1 we have

$$\mu \star (L\tau_{1+\kappa}) = L \times (\mu \star 1 \star \tau_\kappa) - (L\mu) \star 1 \star \tau_\kappa.$$

We rewrite the right-hand side using the identities $\mu \star 1 = 1_{n=1}$, $-(L\mu) \star 1 = \Lambda$ as

$$L\tau_\kappa + \Lambda \star \tau_\kappa.$$

Assuming n squarefree and denoting the number of prime divisors of n by $\omega(n)$ the latter expression is

$$(\log n)\kappa^{\omega(n)} + (\log n)\kappa^{\omega(n)-1}$$

with the convention that $0^0 = 1$. Dividing by $\log N$ we obtain the identity

$$\frac{\log n}{\log N}\kappa^{\omega(n)} + \frac{\log n}{\log N}\kappa^{\omega(n)-1} = \sum_{d|n} \mu(d)\tau_{1+\kappa}\left(\frac{n}{d}\right) \left(\frac{\log n}{\log N} - \frac{\log d}{\log N}\right).$$

Now the conclusions of the theorem follow from this identity.

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